# Universal $H-$colorable Graphs Without A Given Configuration

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**Abstract**

For every pair of finite connected graphs $F$ and $H$, and every integer $k$, we construct a universal graph $U$ with the following properties:

1. There is a homomorphism $\pi : U \to H$, but no homomorphism from $F$ to $U$.
2. For every graph $G$ with maximal degree no more than $k$ having a homomorphism $h : G \to H$, but no homomorphism from $F$ to $G$, there is a homomorphism $\alpha : G \to U$, such that $h = \pi \circ \alpha$.

Particularly, this solves a problem presented in [1] and [2] regarding the chromatic number of a universal graph.

## 1 Introduction

For a graph $G$, let $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. Given two graphs $G$ and $H$, a map $f : V(G) \to V(H)$ is a *(graph) homomorphism* from $G$ to $H$ if and only if $\{f(x), f(y)\} \in E(H)$ for all edges $\{x, y\} \in E(G)$. For simplicity, $f : G \to H$ will denote that $f$ is a homomorphism from $G$ to $H$, and $G \not\to H$ means that no homomorphism exists from $G$ to $H$.

Homomorphisms extend the notion of coloring, as $\chi(G) \leq k$ if and only if $G \to K_k$. More generally, we say that $G$ is *$H$-colorable* if there is a homomorphism $G \to H$. If $F \not\to G$, we say that $G$ is *$F$-free*. A graph $U$ is *universal* for a class of graphs $\mathcal{K}$ if, for any graph $G \in \mathcal{K}$, there exists a homomorphism $G \to U$. For example, the complete graph on $k$ vertices is universal for the family of $k$-colorable graphs. Universal graphs were considered earlier in [1] and [2], which established their existence for several classes of graphs, including classes of $F-$free graphs of bounded degree.

When considering $F-$free and $H-$colorable graphs, we can restrict to those graphs $F$ and $H$ that are *cores*, graphs for which every endomorphism is an

automorphism. Indeed, a graph is $F-$free if and only if it is free of the (uniquely determined) core of $F$, and $H-$colorable if and only if it is colorable by the core of $H$, [3]. We have the following theorem from [5]:

**Theorem 1.1** *Let $F$, $H$ be fixed cores. Then for every universal graph $U$ of the class of all $F-$free, $H-$colorable graphs $K_H$, there exists a homomorphism $H \to U$.*

In particular, every universal graph for 3-colorable, triangle-free graphs contains a triangle.

This negative result was complemented by Häggkvist and Hell with a positive result for $F-$free graphs of bounded degree [2]. We say that a graph is *$b-$bounded* if $\Delta(G) \leq b$.

**Theorem 1.2** *Let $F$ be a core, $b$ a positive integer. Then the class $\mathcal{K}_F^b$ of all $b-$bounded $F-$free graphs has a universal $F-$free graph $U_F^b$.*

As the chromatic number of a $b-$bounded graph is $\leq b+1$, all of the graphs in the class $\mathcal{K}_F^b$ have chromatic number $\leq b+1$. Häggkvist and Hell [2] have asked the minimal chromatic number of an universal graph for $K_F^b$. The graphs constructed in [2] all have a large chromatic number. In [1], Galluccio, Hell, and Nešetřil considered the complexity of $H-$coloring a 3-bounded graphs with $\chi(H) \leq 3$, and conjectured that $H-$coloring of 3-bounded graphs is an NP-complete problem for any 3-chromatic triangle-free graph $H$.

In this paper, we answer both of these questions by proving that the class of all $F-$free 3-bounded graphs has a universal $F-$free graph $U$ with $\chi(U) \leq 3$. More generally, our main result, Theorem 3.1, demonstrates the existence of $F-$free, $H-$colorable universal graphs for the family of $b-$bounded, $F-$free, $H-$colorable graphs.

By a result of Johanson [4], we have the following corollary:

**Corollary 1.3** *For every positive integer $b$, there exists a graph $U_b$ with the following properties:*

1. *$U_b$ is triangle-free, and $\chi(U_b) \leq c \cdot \frac{b}{\log b}$ for a constant $c$ independent of $b$.*
2. *$U_b$ is universal for the family of $b-$bounded, triangle-free graphs.*

The paper is organized as follows. In Section 2, we give a proof of a theorem covering a particular case of the main theorem (when $F = K_3 = H$). This construction closely follows the construction given in [1] and [2]. In Section 3, we prove the general theorem, giving a new construction for the above theorem. Section 4 contains some remarks and open problems.

We introduce some notation used for the remainder of the paper. As usual, $[n]$ will denote the set of positive integers $\{1, 2, \ldots, n\}$. If $x$ and $y$ are two vertices in a connected graph $G$, the *distance* $d_G(x, y)$ denotes the number of edges in the shortest path in $G$ joining $x$ and $y$. For a graph $G$ and a non-negative integer $k$, $G^k$ is a graph defined by $V(G^k) = V(G)$ and $E(G^k) = \{\{x, y\} : x, y \in V(G), 1 \leq d_G(x, y) \leq k\}$. For a vertex $v$, $G[v, k]$ denotes the subgraph induced by the vertices $w \in V(G)$ such that $0 \leq d_G(v, w) \leq k$.

If $\Delta(G) = k$, then by induction, for a vertex $v \in G$ and $j \geq 1$, at most $k(k-1)^{j-1}$ points have distance $j$ from $v$. Putting $X_k(d) = 1 + \sum_{j=1}^{d} k(k-1)^{j-1}$, it follows that $|V(G[v,d])| \leq X_k(d)$.

## 2 $k-$colorable, triangle-free universal graphs for $k-$bounded, triangle-free graphs

**Theorem 2.1** *Given a positive integer $k$, let $\mathcal{F}$ be the family of $k-$bounded, triangle-free graphs. There exists a universal graph $U$ such that:*

1. *$\chi(U) \leq k$.*
2. *For every $G \in \mathcal{F}$, there exists a homomorphism $\alpha : G \to U$.*
3. *$U$ is triangle-free.*

**Proof:** Let $G \in \mathcal{F}$. Assume without loss of generality that $G$ is k-regular. Since $\chi(G) \leq k$, we know there exists a proper coloring of $G$, $c_1 : V(G) \to [k]$. For any vertex in $G^3$, the degree is at most $X_k(3) - 1$, so there exists a proper coloring of $G^3$, $c_2 : V(G^3) = V(G) \to [X_k(3)]$. Putting $X = [k] \times [X_k(3)]$, we can define a map $c : V(G) \to X$ by $c(x) = (c_1(x), c_2(x))$. Let $\pi_1 : X \to [k]$ be the projection onto the first coordinate $\pi_1(c_1, c_2) = c_1$.

The universal graph $U$ is defined by:

$$V(U) = X \times \binom{X}{k}$$

$$E(U) = \{\{(x,A),(x',A')\} : A \cap A' = \emptyset, \; x \in A', x' \in A, \; \pi_1(x) \neq \pi_1(x')\}.$$

1. $\chi(U) \leq k$.

   The map $\pi : V(U) \to [k]$ given by $\pi((x,A)) = \pi_1(x)$ $k-$colors $U$.

2. For every $G \in \mathcal{F}$, there exists a homomorphism $\alpha : G \to U$.

   Put $\alpha(v) = (c(v), \{c(w) : \{v,w\} \in E(G)\})$. If $\{v,w\} \in E(G)$ with $\alpha(v) = (x,A)$, $\alpha(w) = (x',A')$, then:

   a. $A \cap A' = \emptyset$: Because $G$ is triangle-free, $v$ and $w$ have no common neighbors. No neighbor of $v$ can receive the same $X_k(3)-$color as a neighbor of $w$, because neighbors of $v$ and $w$ are at most distance three from one another. Thus, any members of $A$ and $A'$ differ in their $x_2-$coordinate.

   b. $x \in A'$, $x' \in A$: Because $\{v,w\} \in E(G)$, $x' = c(w) \in A$ and $x = c(v) \in A'$.

   c. $\pi_1(x) \neq \pi_1(x')$: $\pi_1(x) = c_1(v)$ and $\pi_1(x') = c_1(w)$, and $c_1$ properly $k-$colors $G$, hence $\pi_1(x) \neq \pi_1(x')$.

3. $U$ is triangle-free.

   Suppose that $(x, A)$, $(x', A')$, and $(x'', A'')$ are three vertices of $U$ that form a triangle. Then $x \in A'$ and $x \in A''$, contradicting $A' \cap A'' = \emptyset$.

Although [1] conjectured that for triangle-free graphs $H$ with chromatic number 3, the $H-$coloring problem for 3$-$bounded graphs is NP-complete, setting $k = 3$ in the above theorem proves this conjecture false: All 3$-$bounded, triangle-free graphs are $H-$colorable, and all 3$-$bounded graphs containing a triangle are not.

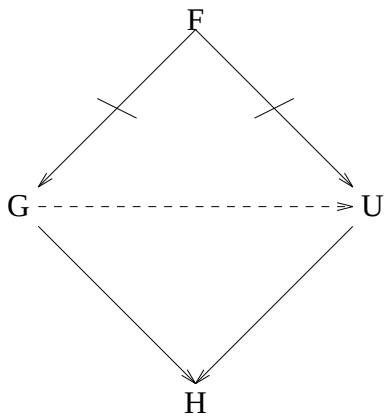
# 3 $F-$free universal graphs for $H-$colorable, $F-$free graphs

The preceding construction mapped triangle-free, $k-$bounded graphs into a universal graph by describing a vertex's neighborhood with the colors used by a coloring of $G^3$. The condition $A \cap A' = \emptyset$ sufficed to avoid triangles in the universal graph. In order to avoid more complicated graphs, we replace the coloring of $G^3$ with a labelled copy of the induced subgraph formed by a large neighborhood around the vertex, which will be $F-$free. Edges in the universal graph connect only the neighborhoods that might be had by adjacent vertices in $F-$free graphs. This idea allows us to strengthen the preceding theorem:

**Theorem 3.1** *Let $F$, $H$ be finite connected graphs, $b$ a positive integer. Then there exists a universal graph $U$ with the following properties:*

1. *$U$ is $H-$colorable.*
2. *For every $b-$bounded, $F-$free, $H-$colorable graph $G$, there exists a homomorphism $G \to U$ that preserves the $H-$coloring of $G$.*
3. *$U$ is $F-$free.*

This result can be expressed using a commutative diagram:



**Proof:** Throughout this proof, we assume that given a $K_n$, its vertices are labelled by $[n]$. Let $m = |V(F)|$, and let $\mathcal{S}$ be the set of all connected subgraphs $S$ of $K_{X_k(m)}$ such that $F \not\to S$.

We will now describe the universal graph $U$.

$$\begin{aligned} V(U) &= \mathcal{S} \times V(K_{X_k(m)}) \times V(H) \\ E(U) = \{\{(S,v,n),(S',v',n')\} :& \{n,n'\} \in E(H), \\ & \{v,v'\} \in E(S) \cap E(S'), \\ & S[v,m-1] = S'[v,m-1], \\ & S[v',m-1] = S'[v',m-1]\} \end{aligned}$$

Here, graph equalities mean not only that the two graphs are isomorphic, but that the labels on the vertices and edges in each graph are identical.

1. $U$ is $H-$colorable.

   The map $\pi : U \to H$ defined by $\pi((S,v,n)) = n$ is a homomorphism that preserves the $H-$coloring of $G$.

2. For every $b-$bounded, $F-$free, $H-$colorable graph $G$, there exists a homomorphism $G \to U$ that preserves the $H-$coloring of $G$.

   Let $G$ be a graph that satisfies the conditions above. Because $G^m$ is $X_k(m)-$colorable, we can fix a homomorphism $c : G \to K_{X_k(m)}$. If $v \in V(G)$ and $r \leq m$, then $c$ is injective on $G[v,r]$, so $c(G[v,r])$ is isomorphic to $G[v,r]$. In particular, $F \not\to c(G[v,r])$, since $c(G[v,r])$ is isomorphic to a subgraph of $G$.

   Because $F \not\to c(G[v,m])$, we can define $\alpha : V(G) \to V(U)$ by $\alpha(x) = (c(G[v,m]), c(x), h(x))$. The map $\alpha$ preserves the $H-$coloring of $G$, namely $h = \pi \circ \alpha$. We claim that $\alpha$ is a homomorphism from $G$ to $U$. Suppose $\{x,x'\} \in E(G)$. Because $h$ is a homomorphism, $\{h(x),h(x')\} \in E(H)$. Since $G[x,m]$ and $G[x',m]$ are induced subgraphs of $G$, both containing $x$ and $x'$, and $c$ is a homomorphism, $\{c(x),c(x')\} \in E(c(G[x,m])) \cap E(c(G[x',m]))$. From $G[x,m-1] \subseteq G[x',m]$ and $G[x',m-1] \subseteq G[x,m]$, we obtain:

   $$\begin{aligned} c(G[x,m])[c(x),m-1] &= c(G[x,m-1]) \\ &= c((G[x',m])[x,m-1]) \\ &= c(G[x',m])[c(x),m-1]. \end{aligned}$$

   Similarly, $c(G[x',m])[c(x'),m-1] = c(G[x,m])[c(x'),m-1]$. Hence, $\{\alpha(x),\alpha(x')\} \in E(U)$.

3. $U$ is $F-$free.

Let $V(F) = \{x_1, \ldots, x_m\}$. Suppose that a homomorphism $\beta : F \to U$ exists, and $\beta(x_i) = (S_i, v_i, n_i)$. We claim that the map $\beta^* : F \to S_1$ defined by $\beta^*(x_i) = v_i$ is a homomorphism, contradicting the condition on $V(U)$ that $F \not\to S$ for all $S \in \mathcal{S}$.

If $\{x_i, x_j\} \in E(F)$, $d_F(x_1, x_i) \leq m - 1$ because $|V(F)| = m$. Since $\{\beta(x_i), \beta(x_j)\} \in E(U)$, $\{v_i, v_j\} \in E(S_i) \cap E(S_j)$.

If $\{x_{k_1} = x_1, x_{k_2}, \ldots, x_{k_p} = x_i\}$ is the shortest path in $F$ from $x_1$ to $x_i$, then $\{v_i, v_j\} \in E(S_{k_p}[v_{k_p}, 1])$. Since $\{v_{k_{p-1}}, v_{k_p}\} \in E(S_{k_p}[v_{k_p}, 1])$,

$$\{v_i, v_j\} \in E(S_{k_p}[v_{k_{p-1}}, 2]) = E(S_{k_{p-1}}[v_{k_{p-1}}, 2]).$$

By induction, we can show that for any $q$, $1 \leq q \leq p < m$, that

$$\{v_i, v_j\} \in E(S_{k_{p-(q-1)}}[v_{k_{p-(q-1)}}, q]).$$

Taking $q = p$,

$$\{v_i, v_j\} \in E(S_{k_1}[v_{k_1}, p]) = E(S_1[v_1, p]) \subseteq E(S_1).$$

Thus, $\beta^*$ is a homomorphism from $F$ to $S_1$, producing a contradiction and completing the proof.

# 4 Remarks and Open Problems

The theorems above suggest that one needs some restrictive condition to guarantee the existence of an universal graph. In the theorems above, the $b-$boundedness condition in the graph family is essential. The following problems appear to be open.

- Does there exist a triangle-free graph $U$ which is universal for every triangle-free planar graph?
- Does there exist a triangle-free planar graph $U$ which is universal for all 3-bounded planar graphs?

We can observe that there is no triangle-free planar graph $U$ which is universal for all triangle-free planar graphs. To see this, consider "sunflower graphs" $Sun_n$. To construct $Sun_n$, begin with a $K_{1,n}$, where the lone vertex is labelled $x$ and the $n$ vertices are labelled by $[n]$. Vertices $i$ and $i+1$ are joined by edge-disjoint paths of length 3, as well as vertices $n$ and 1. Then, the neighbors of each vertex in $[n]$ (not including $x$) are joined by edge-disjoint paths of length three. The sunflower graph $Sun_n$ has $5n+1$ vertices and $7n$ edges. The graphs of $Sun_n$ are shown below for $n = 1, 2, 3$, and 6. One can check that there is

a homomorphism $Sun_n \to Sun_{n'}$ if and only if $n'|n$. One can also check that any planar homomorphic image of $Sun_n$ must have a vertex of degree $n'$, where $n'|n$ and $n' > 1$. If $n$ is prime and exceeds $\Delta(U)$, then $Sun_n \not\to U$.

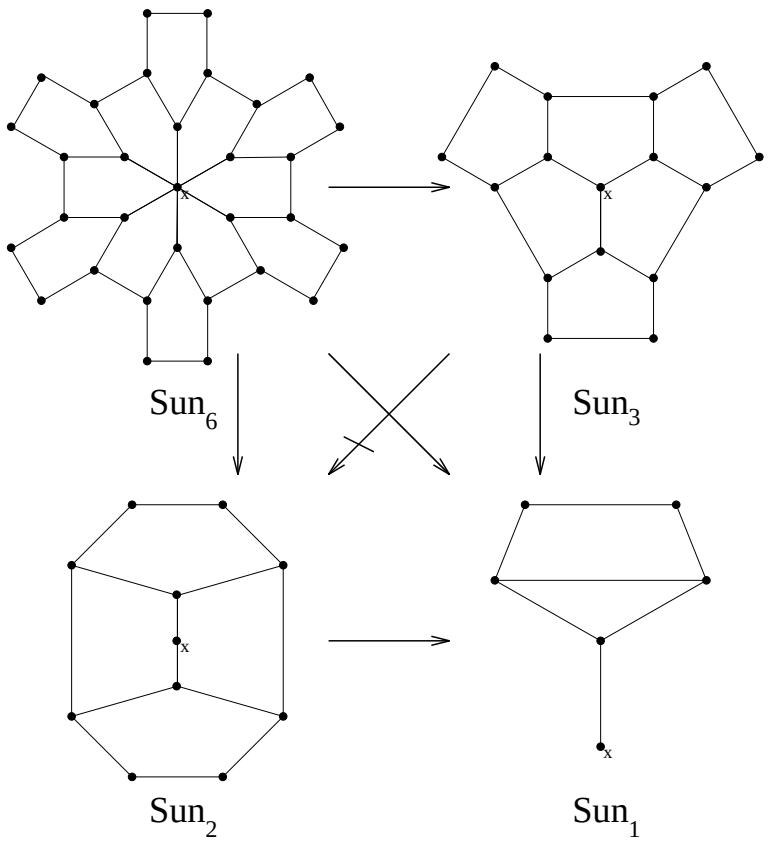



Figure 1: Sunflower graphs for $n = 1, 2, 3$, and 6. Note that although $Sun_1$ can be constructed, it is not triangle-free. Arrows denote that homomorphisms exist from one graph to the other; slashed arrows indicate that no homomorphism exists.

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